Homework 7

PHYS798C Spring 2024
Due Thursday, 4 April, 2024

1 Vortex Screening Currents

James Annett, *Superconductivity, Superfluids and Condensates*, Exercise 3.3.
(a) A z-oriented vortex in a superconductor can be modeled as a magnetic field profile $B_z(r)$, and having a cylindrical core of normal metal of radius ξ_0. Use $\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{B}\right) = -\overrightarrow{B}/\lambda^2$ and the expression for the curl in cylindrical polar coordinates (r, θ, z) to show that the magnetic field $B_z(r)$ outside of the core obeys the Bessel equation: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_z}{dr}\right) = \frac{B_z}{\lambda^2}$.

(b) For small r, obeying $\xi_0 < r << \lambda$, the right hand-side of the Bessel equation in (a) can be approximated by zero. Show that this approximation leads to $B_z(r) = a\, ln(r) + b$, where a and b are unknown constants.

(c) Show that the current corresponding to the field $B_z(r)$ found in (b) is equal to $\vec{J}_s = -\frac{a}{\mu_0 r}\hat{\theta}$ similar to the superfluid current in a ^{4}He vortex. Hence find the vector potential $\vec{A}$ and find a as a function of the magnetic flux enclosed by the vortex, Φ.

(d) For larger values of r ($r \sim \lambda$ and above) assume that we can approximate the Bessel equation from (a) by: $\frac{d}{dr}\left(\frac{dB_z}{dr}\right) = \frac{B_z}{\lambda^2}$. Hence show that $B_z \sim e^{-r/\lambda}$ for large r.

(e) The large-r solution given in part (d) is not exactly the correct asymptotic form of the solution. For large values of r assume that $B_z \sim r^p e^{-r/\lambda}$ and hence show that the correct exponent is $p = -1/2$.

2 Vortex Energy Density

The energy density near the core of a superconducting vortex is expected to be nearly all kinetic. Hence the radius of the core can be estimated assuming the excess kinetic energy unbinds the Cooper pairs there. The kinetic energy density w_{kin} is given by

$$w_{kin} = \frac{1}{2}\Lambda J_s^2$$

(a) Using the expression for $\vec{J}_s$ from Problem 1, show that near the core

$$w_{kin} = \frac{\Phi_0^2}{8\pi^2\mu_0\lambda^2\xi^2}.$$

This expression can be written as $w_{kin} = \mu_0 H_c^2$. Explain qualitatively why such a result is expected.

(b) Show that this expression can also be written as

$$w_{kin} = \frac{\pi^2}{4}\left(\frac{n^*}{\frac{1}{2}m^* v_F^2}\right)\Delta^2.$$

The term in parentheses is proportional to the average number of particles per unit energy, in other words the electronic density of states.

3 Energy per unit length of the vortex

Suppose that any supercurrent flow corresponds to an effective superfluid flow velocity $\overrightarrow{v}$ of the electrons, where $\vec{J}_s = -en_s\overrightarrow{v}$. Assume that the corresponding kinetic energy is $\frac{1}{2}mv^2n_s$ per unit volume. Hence, using the results from the "Vortex Screening Currents" problem parts (c) and (d), show that the total energy per unit length E of a vortex line is roughly of order $E = \frac{\Phi^2}{4\pi\mu_0\lambda^2}\ln\left(\frac{\lambda}{\xi_0}\right)$.